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ENTIRE SOLUTIONS OF CERTAIN PARTIAL DIFFERENTIAL EQUATIONS IN $\mathbf{C}^n$

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ABSTRACT

We describe entire solutions in $\mathbf{C}^n$ of non-linear partial differential equations of the form $(\partial w/\partial z_j)^k = f(w)$, where f is a meromorphic function in the complex plane and k is a positive integer.

This paper is concerned with the description of entire analytic solutions to first-order non-linear partial differential equations of the form $(\partial w/\partial z_j)^k = f(w)$ in $\mathbf{C}^n$, where f is a meromorphic function in the complex plane $\mathbf{C}$ and k is a positive integer. When $n = 1$, the equations are related to Malmquist–Yosida type ordinary differential equations of the form $(y')^k = R(z, y)$, where R is a rational function of $z \in \mathbf{C}$ and y. Although the order of growth of a transcendental solution to the ordinary differential equations can be described (either zero, a positive integral multiple of $\frac{1}{2}$, or a positive integral multiple of $\frac{1}{3}$; see the main result in [BK]), no descriptions of the solutions themselves are known. It is natural and desirable to describe solutions for such differential equations, even though it might be hard or impossible to do so in the most general form. When $k = 1$, the problem on partial differential equations of the form $\partial w/\partial z_j = f(w)$ was considered in [LS] for its relations with some other questions in complex analysis, and the description of entire solutions was given by utilizing an elementary characterization of polynomials. It was shown in [LS] that entire solutions F of the partial differential equation $\partial w/\partial z_j = f(w)$,

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where f is a meromorphic function in $\mathbf{C}$, are either of the form $F = az_j + \alpha$ or of the form $F = a + e^{bz_j+\alpha}$, where a, b are constants and α is an entire function in $z' = (z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) \in \mathbf{C}^{n-1}$. This result is, however, no longer true for partial differential equations $(\partial w/\partial z_j)^k = f(w)$ when $k \geq 2$. For example, $\sin(z_1 + z_2 + \cdots + z_n)$ is an entire solution of the partial differential equation $(\partial w/\partial z_j)^2 = 1 - w^2$ in $\mathbf{C}^n$, which is not of any of the above forms. The purpose of this paper is to describe entire solutions for partial differential equations $(\partial w/\partial z_j)^k = f(w)$ for arbitrary integers $k \geq 2$. That is, we prove the following

THEOREM 1: *Let F be an entire solution in $\mathbf{C}^n$ of the partial differential equation $(\partial w/\partial z_j)^k = f(w)$, say $j = 1$, where $k \geq 2$ is an integer and f is a meromorphic function in $\mathbf{C}$. Then F is of one (and only one) of the following forms:*

(i) $F = az_1 + \alpha(z_2, z_3, \ldots, z_n)$,

(ii) $F = a + e^{bz_1+\alpha(z_2,z_3,\ldots,z_n)}$,

(iii) $F = a + b\sin(cz_1 + \alpha(z_2, z_3, \ldots, z_n))$,

(iv) $F = a + b(z_1 + \alpha(z_2, z_3, \ldots, z_n))^{k/(k-m)}$,

where a, b, c are complex numbers with $bc \neq 0$, $\alpha(z_2, z_3, \ldots, z_n)$ is an entire function in $\mathbf{C}^{n-1}$, and m $(< k)$ is a positive integer such that $k/(k-m)$ is an integer.

Remark 2: We give examples to show that each form of entire solutions in Theorem 1 can indeed occur. Let $\alpha(z_2, z_3, \ldots, z_n)$ be an arbitrary entire function in $\mathbf{C}^{n-1}$ in all the following examples.

(a) Consider the partial differential equation $(\partial w/\partial z_1)^k = 1$. The function $F = z_1 + \alpha(z_2, z_3, \ldots, z_n)$ is an entire solution of the equation and is of the form (i) in Theorem 1.

(b) Consider the partial differential equation $(\partial w/\partial z_1)^k = (-1)^k w^k$. The function $F = e^{-z_1+\alpha(z_2,z_3,\ldots,z_n)}$ is an entire solution of the equation and is of the form (ii) in Theorem 1.

(c) Consider the partial differential equation $(\partial w/\partial z_1)^2 = 1 - w^2$. Then $w = \sin(z_1 + \alpha(z_2, z_3, \ldots, z_n))$ is an entire solution of the equation. This solution is of the form (iii) in Theorem 1.

(d) Consider the partial differential equation $(\partial w/\partial z_1)^4 = 16w^2$. Then $w = (z_1 + \alpha(z_2, z_3, \ldots, z_n))^2$ is an entire solution of the equation. This solution is of the form (iv) in Theorem 1 with $k = 4, m = 2$.

More generally, let $k \geq 2$ and $m < k$ be two positive integers such that

$k/(k-m)$ is an integer. Consider the partial differential equation

$$(\partial w/\partial z_1)^k = \left(\frac{k}{k-m}\right)^k w^m.$$

Let $F = (z_1 + \alpha(z_2, z_3, \ldots, z_n))^{k/(k-m)}$. Then one can check that the function F is an entire solution of the partial differential equation and is of the form (iv) in Theorem 1.

Unlike in [LS] for the case $k = 1$, in proving Theorem 1 we will employ Nevanlinna theory. We will assume familiarity with basics of the theory, and also basics of one and several complex variables (see, e.g., [BG], [K], [S]).

Proof of Theorem 1: Since F is an entire solution of the partial differential equation, we have

$$\left(\frac{\partial F(z)}{\partial z_1}\right)^k = f(F(z)), \quad z \in \mathbf{C}^n. \tag{1}$$

If $\partial F(z)/\partial z_1$ is a constant, the function F is already of the form (i) in the theorem. Thus, we assume that $\partial F(z)/\partial z_1$ (and thus F) is not a constant in the following proof.

Suppose that f is transcendental. Then the non-constant entire function F must be also transcendental, since otherwise the left-hand side of (1) would be a polynomial but the right-hand side of (1) would be transcendental, which is impossible. We then use the following theorem in [CLY]: If f is a transcendental meromorphic function in $\mathbf{C}$ and g is a transcendental entire function in $\mathbf{C}^n$, then

$$\lim_{r\to\infty} \frac{T(r, f(g))}{T(r,g)} = +\infty,$$

where $T(r,h)$ denotes the Nevanlinna characteristic function of a meromorphic function h in $\mathbf{C}^n$. Thus, we obtain

$$\frac{T(r, f(F))}{T(r,F)} \to \infty$$

as $r \to \infty$. On the other hand, we have that $T(r, \partial h/\partial z_j) = O\{T(r,h)\}$ for any meromorphic function h in $\mathbf{C}^n$ outside a set of r of finite Lebesgue measure (see, e.g., [S], [V]). Therefore, it follows from (1) that

$$T(r, f(F)) = T(r, (\partial F/\partial z_1)^k) = kT(r, \partial F/\partial z_1) = O\{T(r,F)\}$$

outside a set of finite Lebesgue measure, a contradiction to the above limit. This shows that f must be a rational function.

We notice that f has at most one pole in $\mathbf{C}$. Otherwise, by the Picard theorem, F has at most one finite Picard value and thus would take one of the poles of f infinitely many times, and then the right-hand side of (1) would not be an entire function, a contradiction.

We next show that f actually has no poles. Assume to the contrary that f has one pole $a \in \mathbf{C}$. We can then write f in the following form:

$$f(w) = \frac{1}{(w-a)^m} g(w)$$

for $w \neq a$ in $\mathbf{C}$, where $m > 0$ is an integer, and g is a polynomial in $\mathbf{C}$ with $g(a) \neq 0$. By (1), $F(z) \neq a$ for any $z \in \mathbf{C}^n$. Hence we have that $F(z) = a + e^{q(z)}$, $z \in \mathbf{C}^n$, where q is an entire function in $\mathbf{C}^n$, which is non-constant since F is non-constant. By (1) again, we obtain that

$$e^{kq(z)}(\partial q(z)/\partial z_1)^k = \frac{g(a+e^{q(z)})}{e^{mq(z)}},$$

or

$$\left(\frac{\partial q(z)}{\partial z_1}\right)^k = \frac{g(a+e^{q(z)})}{e^{(m+k)q(z)}} := \alpha(q(z)), \tag{2}$$

where $\alpha(w) = g(a+e^w)/e^{(m+k)w}, w \in \mathbf{C}$, is an entire function in the complex plane. The equality (2) has the same form as (1) (comparing α in (2) with f in (1)). Thus, we see that α must be a polynomial by the same argument as above for proving f to be rational. Then we can see that α must be a constant, since otherwise $\alpha = g(a+e^w)/e^{(m+k)w}$ will be a non-constant periodic function and cannot be a polynomial. We thus obtain from (2) that $\partial q(z)/\partial z_1 = c$, a constant. Integrating this equality yields that $q(z) = cz_1 + \alpha(z_2, z_3, \ldots, z_n)$ and so

$$F = a + e^{cz_1 + \alpha(z_2, z_3, \ldots, z_n)},$$

where α is an entire function in $(z_2, z_3, \ldots, z_n)$. Clearly, $c \neq 0$, since $\partial F(z)/\partial z_1$ is not a constant, as assumed in the beginning. It would be tempting to conclude that F is of the form (ii) in the theorem; however, if we substitute the above form for F into the original equation, we obtain

$$(ce^{cz_1+\alpha(z_2,z_3,\ldots,z_n)})^k = f(a + e^{cz_1+\alpha(z_2,z_3,\ldots,z_n)})$$

or

$$(f(w) - c^k(w-a)^k) \circ (a + e^{cz_1+\alpha(z_2,z_3,\ldots,z_n)}) \equiv 0,$$

which implies that $f(w) = c^k(w-a)^k$, $w \in \mathbf{C}$, a contradiction to the assumption that f has a pole. Thus, we have showed that f cannot have any poles. That is, f is a polynomial in $\mathbf{C}$, which is non-constant since $\partial F(z)/\partial z_1$ is non-constant.

We can thus write (1) as

$$(3)\quad \left(\frac{\partial F(z)}{\partial z_1}\right)^k = c(F(z)-a_1)^{m_1}(F(z)-a_2)^{m_2}\cdots(F(z)-a_t)^{m_t}, \quad z \in \mathbf{C}^n,$$

where $t \geq 1$ is an integer, a_j's are distinct complex numbers, m_j's are positive integers, and $c \neq 0$ is a complex number. It is easy to see that

$$(4)\qquad \sum_{j=1}^{t} m_j \leq k,$$

since by (3),

$$\left(\sum_{j=1}^{t} m_j\right) m(r,F) \leq m(r,(\partial F(z)/\partial z_1)^k) + O(1)$$
$$\leq km(r,F) + o\{m(r,F)\},$$

outside a set of r of finite Lebesgue measure (see, e.g., [V], [S]), where $m(r,h)$ denotes the proximity function of a meromorphic function h. Since $\partial F/\partial z_1 \not\equiv 0$, we may choose complex numbers $\zeta_2, \ldots, \zeta_n$ such that $F(z_1, \zeta_2, \ldots, \zeta_n)$ is not a constant function in z_1. By (3), we have

$$(5)\qquad \left(\frac{\partial F(z_1,\zeta_2,\ldots,\zeta_n)}{\partial z_1}\right)^k = c\prod_{j=1}^{t}(F(z_1,\zeta_2,\ldots,\zeta_n)-a_j)^{m_j}.$$

If $F(z_1,\zeta_2,\ldots,\zeta_n) - a_j$, $z_1 \in \mathbf{C}$, has a zero, say w_j with multiplicity l_j, then the multiplicity l_j is at least 2, since $\partial F(z_1,\zeta_2,\ldots,\zeta_n)/\partial z_1$ also vanishes at the zero w_j, in view of (5). (It is worth noting here that this fact on multiplicities cannot follow directly from (3) for a zero of $F(z_1,z_2,\ldots,z_n) - a_j$ in $\mathbf{C}^n$, since the multiplicity of a zero of $F(z_1,z_2,\ldots,z_n) - a_j$ might be lower than or equal to the one for its partial derivative $\partial F(z_1,z_2,\ldots,z_n)/\partial z_1$. This phenomenon never happens in one complex variable.) By comparing the multiplicities of the zero w_j of both sides of (5), we have that $k(l_j - 1) = m_j l_j$ and so

$$(6)\qquad m_j = k(1 - 1/l_j) \geq k/2.$$

We assert that $t \leq 2$. In fact, by the Picard theorem there exists at most one a_j such that $F(z_1,\zeta_2,\ldots,\zeta_n) - a_j$ has no zeros. Thus, if $t \geq 3$, then there are at least two a_js such that $F(z_1, \zeta_2, \ldots, \zeta_n) - a_j$ has zeros. We then deduce, by (6) and the fact that $m_j \geq 1$, that $\sum_{j=1}^{t} m_j \geq k/2 + k/2 + 1 = k+1$, a contradiction to (4). We next consider two cases: $t = 1$ and $t = 2$.

CASE 1: $t = 2$. In this case, since $F(z_1, \zeta_2, \ldots, \zeta_n)$ has at most one Picard value, we may assume that one of a_1 and a_2, say a_2, is not a Picard value of $F(z_1, \zeta_2, \ldots, \zeta_n)$, which implies, by (6), that $m_2 \geq k/2$. We can then write the equality (3) as

$$\left(\frac{\partial F(z)}{\partial z_1}\right)^k = c(F(z) - a_1)^{m_1}(F(z) - a_2)^{m_2}, \tag{7}$$

where $m_2 \geq k/2$. We discuss two subcases in the following:

CASE 1(i): There exists a point $(w_2, w_3, \ldots, w_n) \in \mathbf{C}^{n-1}$, such that the one variable function $F(z_1, w_2, w_3, \ldots, w_n) - a_1$ in z_1 has a zero in $\mathbf{C}$. In this case, by the above arguments for proving (6), we know that $m_1 \geq k/2$. Recall that $m_2 \geq k/2$, and that $m_1 + m_2 \leq k$ by (4). We thus must have that $m_1 = m_2 = k/2$. Therefore, the equality (7) reduces to

$$\left(\frac{\partial F(z)}{\partial z_1}\right)^2 = b(F(z) - a_1)(F(z) - a_2), \tag{8}$$

where $b \neq 0$ is a constant. By completing the square for the right-hand side of (8), we can change (8) into

$$\left(\frac{\partial F(z)}{\partial z_1}\right)^2 + b_1^2(F(z) - b_2)^2 = b_3^2,$$

where b_j's are constants satisfying

$$b_1^2 = -b \neq 0, \quad b_2 = \frac{a_1 + a_2}{2}, \quad b_3^2 = b\Big(a_1 a_2 - \Big(\frac{a_1 + a_2}{2}\Big)^2\Big) \neq 0.$$

It then follows that

$$[\partial F(z)/\partial z_1 + ib_1(F(z) - b_2)][\partial F(z)/\partial z_1 - ib_1(F(z) - b_2)] = b_3^2 \neq 0.$$

Since each factor on the left-hand side of the above equality does not vanish in $\mathbf{C}^n$, there exists an entire function h in $\mathbf{C}^n$ such that

$$\partial F(z)/\partial z_1 + ib_1(F(z) - b_2) = b_3 e^{ih(z)}$$

and then

$$\partial F(z)/\partial z_1 - ib_1(F(z) - b_2) = b_3 e^{-ih(z)},$$

which yields

$$\partial F(z)/\partial z_1 = b_3 \frac{e^{ih}+e^{-ih}}{2} = b_3 \cos h$$

and

$$b_1(F(z)-b_2) = b_3 \frac{e^{ih}-e^{-ih}}{2i} = b_3 \sin h.$$

From the second equality, we have that

$$\frac{\partial F(z)}{\partial z_1} = \frac{b_3}{b_1}\frac{\partial h}{\partial z_1}\cos h,$$

which implies that h is non-constant since $\partial F(z)/\partial z_1$ is non-constant. Combining this with the first equality yields that $\partial h(z)/\partial z_1 = b_1$. We then have $h(z) = b_1 z_1 + \alpha(z_2, z_3, \ldots, z_n)$, where α is an entire function in $\mathbf{C}^{n-1}$, and thus

$$F = b_2 + \frac{b_3}{b_1}\sin(b_1 z_1 + \alpha(z_2, z_3, \ldots, z_n)),$$

which is of the form (iii) in Theorem 1.

CASE 1(ii): For any $(w_2, w_3, \ldots, w_n) \in \mathbf{C}^{n-1}$, $F(z_1, w_2, w_3, \ldots, w_n) - a_1$ has no zeros for $z_1 \in \mathbf{C}$. This clearly implies that the function $F(z_1, z_2, \ldots, z_n) - a_1$ has no zeros for all $z = (z_1, z_2, \ldots, z_n) \in \mathbf{C}^n$. Therefore, we can write $F(z) = a_1 + e^{g(z)}$, $z \in \mathbf{C}^n$, where g is a non-constant entire function in $\mathbf{C}^n$ in view of the fact that F is non-constant. By (7), we obtain

$$\left(\frac{\partial g(z)}{\partial z_1}\right)^k = ce^{(m_1-k)g(z)}(a_1 - a_2 + e^{g(z)})^{m_2} := \beta(g(z)),$$

where $\beta(w) := ce^{(m_1-k)w}(a_1 - a_2 + e^w)^{m_2}$ is an entire function in the complex plane. The above equality is again of the same form as (1). We see that β must be a polynomial (comparing β with f in (1)). Then we can further see that β is actually a constant, since otherwise β will be a non-constant periodic function and cannot be a polynomial. We thus obtain that $\partial g(z)/\partial z_1 = b$, a constant. Integrating this equality yields $g(z) = bz_1 + \alpha(z_2, z_3, \ldots, z_n)$ and so

$$F = a_1 + e^{bz_1 + \alpha(z_2, z_3, \ldots, z_n)},$$

where α is an entire function in $\mathbf{C}^{n-1}$. Clearly, $b \neq 0$ since $\partial F(z)/\partial z_1$ is non-constant. This solution F is of the form (ii) in the theorem. Next we consider

CASE 2: $t = 1$. In this case, the equality (3) can be written as

$$\left(\frac{\partial F(z)}{\partial z_1}\right)^k = c(F(z) - a_1)^{m_1}, \tag{9}$$

where $m_1 \leq k$ by (4). If $m_1 = k$, then F is clearly of the form (ii) in the theorem. We assume that $m_1 < k$ in the following proof.

Denote $z' := (z_2, z_3, \ldots, z_n) \in \mathbf{C}^{n-1}$. We may expand $F(z_1, z')$, as an entire function in z_1, in the following Taylor series:

$$(10) \qquad F(z_1, z') = c_0(z') + c_1(z')z_1 + \cdots + c_m(z')z_1^m + \cdots.$$

We claim that there are only finitely many c_m that are not zero, i.e., F is a pseudo-polynomial in z_1. Suppose that there are infinitely many c_m that are not identically zero in $\mathbf{C}^{n-1}$. Then we can take a point $z' \in \mathbf{C}^{n-1}$ such that $F(z_1, z')$ is a transcendental entire function in z_1. (Such a z' does exist, since the zero sets of those non-zero functions c_m are thin sets and thus have measure zero in $\mathbf{C}^{n-1}$; and we can take a z' outside all those zero sets.) Since $F(z_1, z') - a_1 \not\equiv 0$, we may take an analytic branch $\log w$ of the logarithm such that

$$(F(z_1, z') - a_1)^{\frac{m_1}{k}} := e^{\frac{m_1}{k} \log F(z_1, z')}$$

is well defined and analytic in z_1 in a domain $G \subset \mathbf{C}$. We can then write (9) as

$$(11) \qquad \partial F(z_1, z')/\partial z_1 = b(F(z_1, z') - a_1)^{m_1/k}$$

for $z \in G$, where b is a constant satisfying $b^k = c$. Integrating (11) with respect to z_1 yields

$$(F(z_1, z') - a_1)^{1 - m_1/k} = (1 - m_1/k)(bz_1 + \alpha(z'))$$

or

$$(12) \qquad (F(z_1, z') - a_1)^{k - m_1} = (1 - m_1/k)^k (bz_1 + \alpha(z'))^k$$

for $z \in G$, where $\alpha(z')$ is a constant depending on z'. Note that both sides of (12) are entire functions of z_1 and they coincide on a domain G. By uniqueness, (12) holds in the whole complex plane $\mathbf{C}$. We see that $(F(z_1, z') - a_1)^{k - m_1}$ is a polynomial in z_1, a contradiction to the assumption that $F(z_1, z')$ is transcendental in z_1. We have thus showed that F is a pseudo-polynomial in z_1, i.e.,

$$(13) \qquad F(z_1, z') = c_0(z') + c_1(z')z_1 + \cdots + c_m(z')z_1^m, \quad z = (z_1, z') \in \mathbf{C}^n,$$

where $m \geq 1$ is an integer and $c_m(z') \not\equiv 0$. Substituting it for F in (9), we obtain $k(m-1) = m_1 m$, which implies $m = k/(k - m_1)$ and hence that $k/(k - m_1)$ is an integer. Also, by (13), for each z' outside the zero sets of c_j $(0 \leq j \leq m)$, which are sets of measure zero in $\mathbf{C}^{n-1}$, $F(z_1, z') - a_1$ is a non-zero polynomial in z_1. We can repeat the argument for proving (12) to obtain

$$F(z_1, z') - a_1 = (1 - m_1/k)^{k/(k-m_1)}(bz_1 + \alpha(z'))^{k/(k-m_1)} \tag{14}$$

for z_1 in a domain of $\mathbf{C}$ and thus in the whole plane $\mathbf{C}$ by uniqueness, where $b \neq 0$ is a constant. Since (14) holds for z' outside a set of measure zero in $\mathbf{C}^{n-1}$, by the continuity of F we obtain

$$F(z) = a_1 + (1 - m_1/k)^{k/(k-m_1)}(bz_1 + \alpha(z'))^{k/(k-m_1)} \tag{15}$$

for all $z \in \mathbf{C}^n$, where $\alpha(z')$ is a function in $z' \in \mathbf{C}^{n-1}$. We assert that the function $\alpha(z')$ is an entire function in z'. In fact, for each $2 \leq j \leq n$ we have by (15)

$$\begin{aligned} 0 &= \partial F(z')/\partial \bar{z}_j \\ &= \Big(1 - \frac{m_1}{k}\Big)^{k/(k-m_1)}\Big(\frac{k}{k-m_1}\Big)(bz_1 + \alpha(z'))^{k/(k-m_1)-1}\partial\alpha(z')/\partial\bar{z}_j, \end{aligned} \tag{16}$$

where, as usual,

$$\partial/\partial\bar{z}_j = \frac{1}{2}\Big(\partial/\partial x_j - \frac{1}{i}\partial/\partial y_j\Big) \quad \text{and} \quad z_j = x_j + iy_j.$$

For any given $z' \in \mathbf{C}^{n-1}$, we can take a z_1 such that $bz_1 + \alpha(z') \neq 0$. Then the equality (16) implies that $\partial\alpha(z')/\partial\bar{z}_j = 0$. This means that α is holomorphic at every point z' and thus that α is entire in $\mathbf{C}^{n-1}$. Therefore, F is of the form (iv) in the theorem.

We have thus showed that F is of one of the forms (i)–(iv) in the theorem. Finally, if F is of one of the forms (i)–(iv), it is clear that F cannot be of another form of (i)–(iv). That is, F is of one and only one form of (i)–(iv). This completes the proof of Theorem 1.

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